Review Session for Dr. Lega's Exam I

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Problems we discussed:

1. How are inductive sets used in proofs by induction?

   Idea: Set S to be the set of all natural numbers n such that the statement P_n is true. If P_1 is true, then $1 \in S$. If P_n being true implies P_{n+1} is true, then $n \in S$ implies $n + 1 \in S$, and therefore S is an inductive set. This implies $S = \mathbb{N}$, and so P_n is true for all n.

2. Is $\{\pm q^2 : q \in \mathbb{Q}\}$ dense in $\mathbb{R}$?

   Idea: This set is indeed dense. Let $a, b \geq 0$. By density of rational numbers, there is $q \in (\sqrt{a}, \sqrt{b})$, and therefore $a < q^2 < b$. In the case that $b > 0$ and $a < 0$, apply the previous to $(0, b)$. In the case $a, b < 0$, apply the previous to find some $q \in (\sqrt{|b|}, \sqrt{|a|})$, then $-q^2 \in (a, b)$.

3. For $a, b > 0$, show $q \in (\sqrt{a}, \sqrt{b})$ implies $a < q^2 < b$ (this shows up in the last problem).

   Idea: $a = \sqrt{a}\sqrt{a} < \sqrt{a}q < qq = q^2 < \sqrt{b}q < \sqrt{b}\sqrt{b} = b$.

4. Prove the Cauchy inequality: $ab \leq \frac{a^2+b^2}{2}$.

   Idea: Suppose $ab > (a^2 + b^2)/2$. Then $a^2 - 2ab + b^2 < 0$. On the other hand, $a^2 - 2ab + b^2 = (a - b)^2 \geq 0$, so we get a contradiction.

5. For $a, b, c \geq 0$, show $ab + bc + ca \leq a^2 + b^2 + b^2$.

   Idea: Use the Cauchy inequality to each term.

$$ab + bc + ca \leq \frac{a^2 + b^2}{2} + \frac{b^2 + c^2}{2} + \frac{c^2 + a^2}{2} = a^2 + b^2 + c^2.$$

6. For $a, b, c \geq 0$, show $8abc \leq (a+b)(b+c)(c+a)$.

   Idea: For this problem, we need a little bit of inspiration! This inequality reminds us of the Cauchy inequality, but the powers are different. What can we do? Apply the Cauchy inequality to $\sqrt{a}, \sqrt{b}$, like so:

$$\sqrt{ab} = \sqrt{a}\sqrt{b} \leq \frac{(\sqrt{a})^2 + (\sqrt{b})^2}{2} = \frac{a+b}{2}.$$

   (Note: In this form, the inequality is often referred to as "Arithmetic Mean-Geometric Mean inequality.")

   We want the 2 to be on the other side (to get a power of 8 on the LHS, as in the problem statement), so

$$2\sqrt{ab} \leq a + b.$$

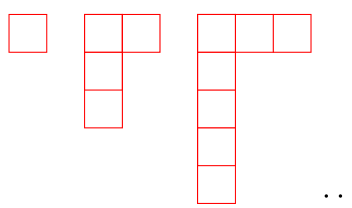
   Finally, apply our modified inequality to $8abc$:

$$8abc = 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ac} \leq (a+b)(b+c)(a+c).$$

7. Show there is no rational number q such that $q^2 = 2/9$.

   Idea: Suppose such a rational number q exited. Then $9q^2 = 2$, or equivalently, $(3q)^2 = 2$. This would imply that $3q$ is a rational square root of 2, a contradiction of a fact we proved earlier in the course!

8. Consider the following sequence of shapes:



   How many total cells are in the first n shapes? Prove your formula using induction.

   Idea: There are $3k - 2$ cells in the $k$the shape. Therefore the total number of cells is

$$\sum_{k=1}^{n} (3k-2) = 3\sum_{k=1}^{n} k - 2\sum_{k=1}^{n} 1 = \frac{3(n+1)n}{2} - 2n.$$

To prove this is true, set the above sum equal to S_n, then show $S_1 = 1$ and $S_{n+1} - S_n = 3$.

Good luck on your exam!