Review Session 5 Problems

Gleb Zhelezov

07 October, 2013

Problems we discussed:

1. Prove $a_n, b_n \to 0$ for $a_n = \frac{\sin(n^2+8n+9)}{n}$ and $b_n \to \frac{n}{n^n}$.

   Idea: For a_n, note $|\sin(x)| \leq 1$. For b_n, note that n grows much slower than 2^n.

2. Suppose $\{x_n\} \geq 0$. Prove that if $x_n \to x$, then $\sqrt{x_n} \to \sqrt{x}$.

   Idea: For $x = 0$, this is easy: Fix $\epsilon^2 > 0$, then for large enough n, $x < \epsilon^2$, from which $\sqrt{x} < \epsilon$. For $x \neq 0$, apply the previous to the sequence $\{|x_n - x|\} \to 0$. Then $\sqrt{x_n} - \sqrt{x} = \sqrt{x_n - x + x} - \sqrt{x} \leq \sqrt{|x_n - x|} + \sqrt{x} - \sqrt{x} = \sqrt{|x_n - x|} \to 0$.

3. Interpret $0.a_1a_2a_3\ldots$ as

$$0.a_1a_2a_3\ldots = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{a_n}{10^n}.$$

   Prove the following controversial equalities: $0.499\ldots = 0.5$, and $0.999\ldots = 1$.

   Idea: Look at the difference between the partial sum and 0.5. The difference gets smaller (on the order of $1/10^N$), so we just need to rewrite this idea in terms of deltas and epsilons.

4. Suppose $\{x_n\}$ is a sequence in $\mathbb{R}$. Prove x_n converges to a is and only if every subsequence of the sequence also converges to a.

   Idea: Write down the definition for convergences, note that subsequences just "skip forward in the sequence," thus eventually converging at a.

The other direction is trivial: $\{x_n\}$ is a subsequence of itself.

5. Suppose $x_0 \in \mathbb{R}$, $x_n = \frac{1+x_{n-1}}{2}$. Prove $x_n \to 1$.

   Idea: Write $|x_n - 1|$, rewrite in terms of x_{n-1}, repeat this idea until you write this difference in terms of x_0, then note that this difference decreases as $n \to \infty$.